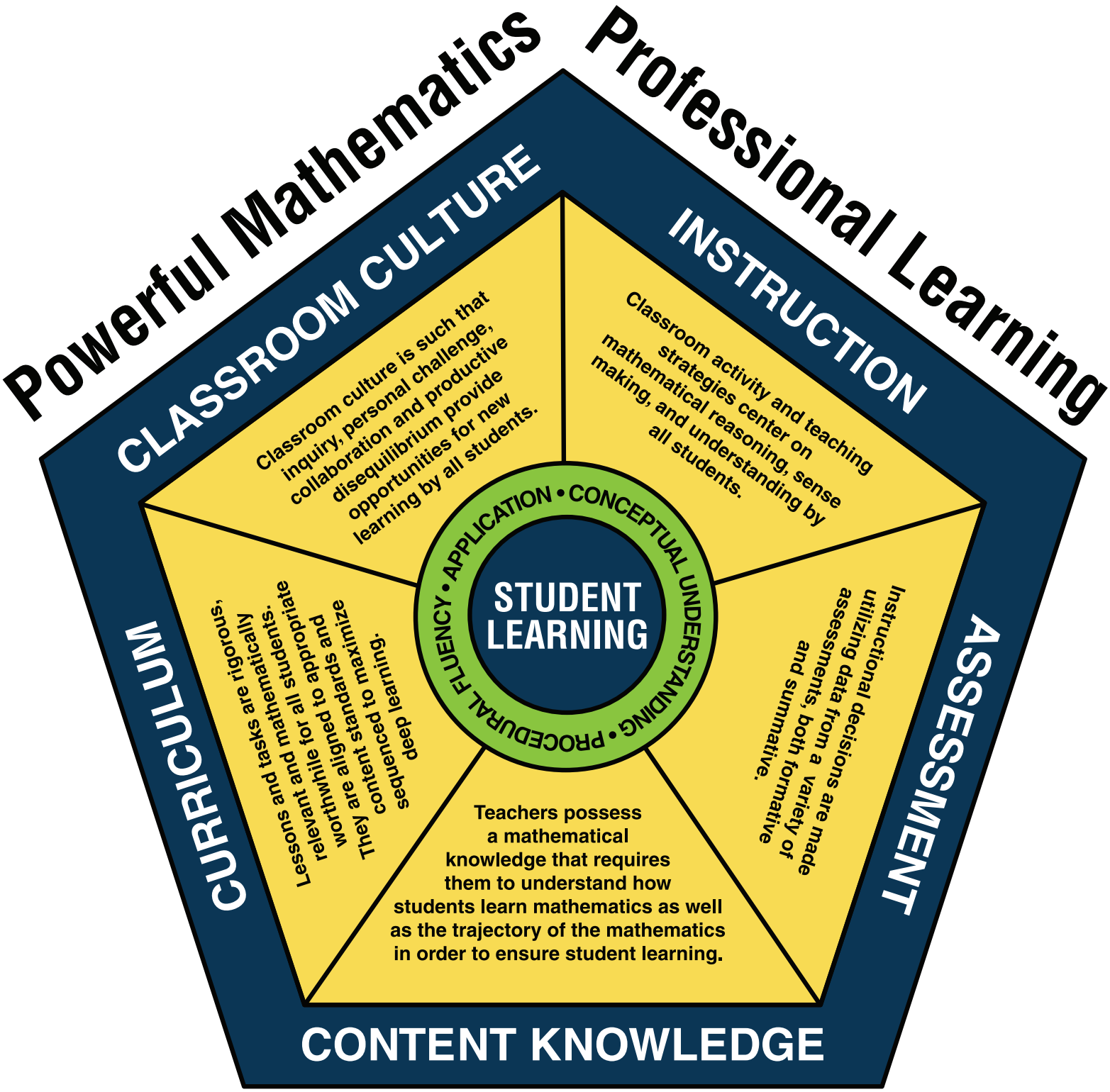
Teaching for Understanding Mathematics Framework



Mathematics can and must be learned by all students.

Table of Contents

Mission/Beliefs/Goals	2
Research and Alignment	3
Purpose and Structure	11
Domain Characteristics	13
Habits of Mind and Interaction	20
Planning Units and Lessons	25
Mathematically Productive Teaching Practices/Strategies	30
Professional Resources	59

HCDE Mission

The Hamilton County Department of Education is committed to the personal and intellectual success of all students.

HCDE Vision

Our community is dedicated to providing a world-class education. We will foster the development of students who are creative thinkers, life-long learners, and productive citizens.

Mission/Vision in Mathematics

To develop mathematically literate students who confidently reason, communicate and value mathematics

Beliefs

- ✓ Mathematics can and must be learned by all students.
- ✓ Intelligence is not fixed and can be grown through effort and continued learning.
- ✓ All students will be taught by a highly effective teacher.
- ✓ Mathematical learning incorporates a balance of conceptual understanding, procedural fluency, and application.
- ✓ Classroom culture must be such that inquiry, personal challenge, collaboration, and productive disequilibrium provide opportunities for new learning by all students.
- ✓ Classroom activity and teaching strategies center on mathematical reasoning, sense making, and understanding by all students.
- ✓ Instructional decisions are made utilizing data from a variety of assessments, both formative and summative.
- ✓ Teachers possess a mathematical knowledge that requires them to understand how students learn mathematics as well as the trajectory of the mathematics in order to ensure student learning.
- ✓ Lessons and tasks are rigorous, relevant, and mathematically worthwhile for all students.
- ✓ Curriculum is aligned to appropriate content standards and sequenced to maximize deep learning.
- ✓ All classrooms will have access to and utilize technology.

Goal

Students will demonstrate mastery in mathematics at or above their grade/course level.

This goal will be accomplished by:

- ✓ students having daily opportunities to apply mathematics and solve contextual (real-world) problems.
- ✓ mathematics instruction involving a balance of conceptual understanding, procedural fluency, and application.
- ✓ teachers using mathematically correct terminology and teaching common mathematical processes K-12.
- ✓ teachers making instructional decisions based on a variety of assessments, both formative and summative.
- ✓ teachers of mathematics having the knowledge necessary to teach the content.
- ✓ collaboration occurring class to class, grade to grade, and school to school.

Framework Research and Alignment

The Teaching for Understanding Mathematics Framework has been designed to fully embrace the research behind how students learn mathematics, the five process standards from the National Council of Teachers of Mathematics (NCTM), the five strands of proficiency from the National Research Council (NRC), fostering a growth mindset and the eight Common Core Standards for Mathematical Practice.

How Students Learn Mathematics

(Teachers Development Group: Mathematically Productive Teaching Routines 2013)

1. **Cognitively demanding mathematical tasks**
2. **Adherence to mathematically productive classroom norms and relationships**
3. **Mathematical discourse that focuses on students' mathematical reasoning, sense making, representations, justifications, and generalizations**
4. **Students' reflection and metacognition about their own and each other's mathematical thinking**
5. **Productive disequilibrium about mathematical ideas and relationships**

NCTM's Five Process Standards

(*Principle and Standards for School Mathematics*)

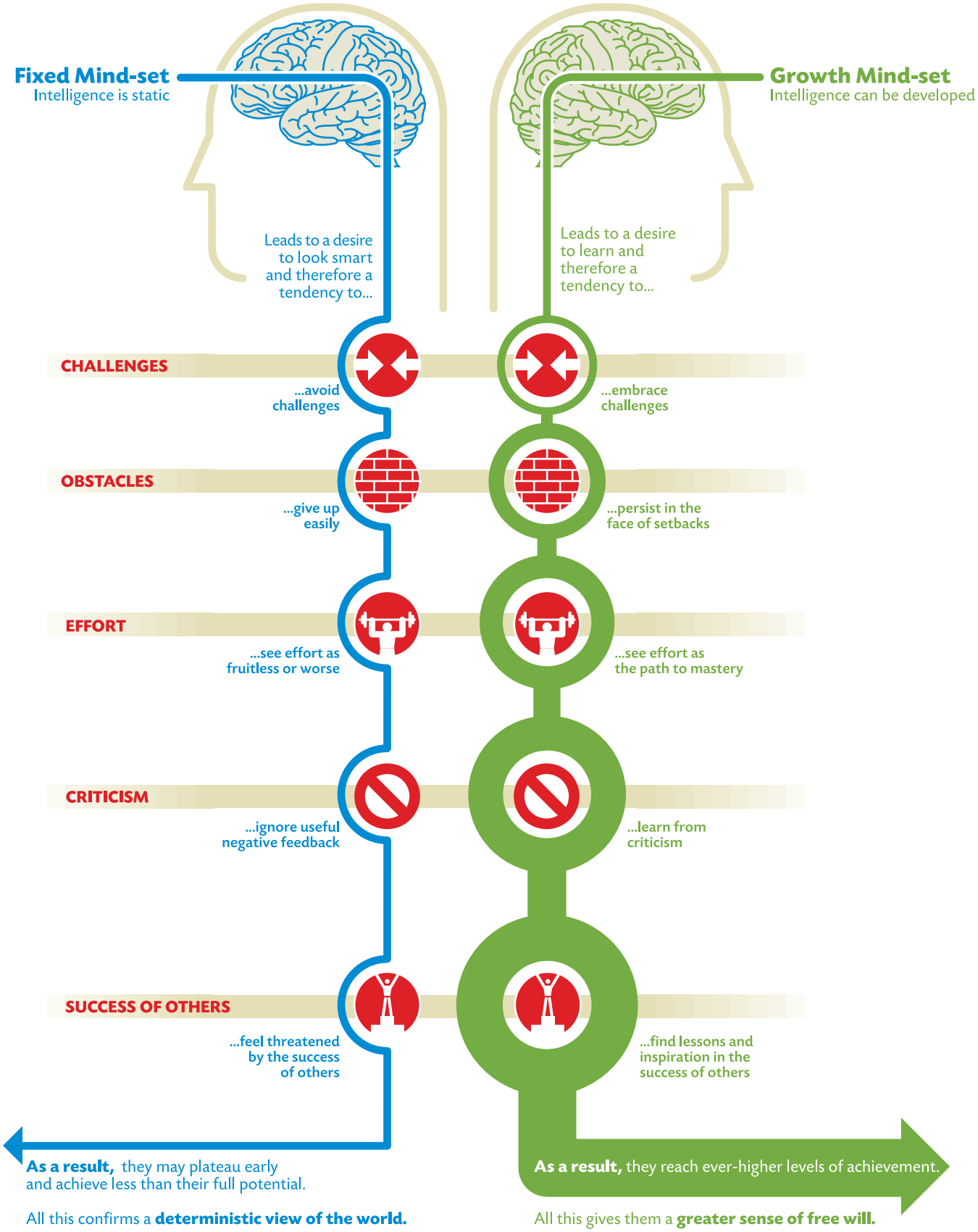
1. **Problem Solving –** Solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving
2. **Reasoning and Proof –** Make and investigate mathematical conjectures; develop and investigate mathematical arguments and proofs; select and use various types of reasoning and methods of proof
3. **Communication –** Communicate mathematical thinking coherently and clearly to peers, teachers, and others; analyze and evaluate the mathematical thinking and strategies of others; use the language of mathematics to express mathematical ideas precisely
4. **Connections –** Understand how mathematics ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics in contexts outside of mathematics
5. **Representation –** Create and use representations to organize, record, and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problems; use representations to model and interpret physical, social, and mathematical phenomena

NRC's Five Strands of Proficiency

(Helping Children Learn Mathematics)

1. **Understanding –** Comprehending mathematical concepts, operations, and relations – knowing what mathematics symbols, diagrams, and procedures mean
2. **Computing –** Carrying out mathematical procedures such as adding, subtracting, multiplying, and dividing numbers flexibly, accurately, efficiently, and appropriately
3. **Applying –** Being able to formulate problems mathematically and to devise strategies for solving them using concepts and procedures appropriately
4. **Reasoning –** Using logic to explain and justify a solution to a problem or to extend from something known to something not yet known
5. **Engaging –** Seeing mathematics as sensible, useful, and doable – if you work at it – and being willing to do the work

Fixed Mindset vs. Growth Mindset



Source: www.isacs.org/misc_files/Mindset%20diagram.pdf

Mindset The New Psychology of Success by Carol S. Dweck

GRAPHIC BY NIGEL HOLMES

Structure of the Common Core Standards for Mathematical Practice

Overarching Habits of Mind of a Productive Mathematical Thinker

1. Make sense of problems and persevere in solving them.
6. Attend to precision.

Reasoning and Explaining

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

Modeling and Using Tools

4. Model with mathematics.
5. Use appropriate tools strategically.

Seeing Structure and Generalizing

7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Tools for the Common Core Standards-Structuring the Mathematical Practices by Bill McCallum - http://commoncoretools.me/2011/03/10/structuring-the-mathematical-practices/

Common Core Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them.**
   Mathematically proficient students:
   - start by explaining to themselves the meaning of a problem and looking for entry points to its solution.
   - analyze givens, constraints, relationships, and goals.
   - make conjectures about the form and meaning of the solution.
   - plan a solution pathway rather than simply jumping into a solution attempt.
   - consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.
   - monitor and evaluate their progress and change course if necessary.
     Older students might, depending on the context of the problem:
     - ✓ transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need.
     - ✓ explain correspondences between equations, verbal descriptions, tables, and graphs.
     - ✓ draw diagrams of important features and relationships, graph data, and search for regularity or trends.

     Younger students might:
     - ✓ rely on using concrete objects or pictures to help conceptualize and solve a problem.
   - check their answers to problems using a different method.
   - continually ask themselves, “Does this make sense?”
   - understand the approaches of others to solving complex problems and identify correspondences between approaches.

2. **Reason abstractly and quantitatively.**
   Mathematically proficient students:
   - make sense of quantities and their relationships in problem situations.
   - bring two complementary abilities to bear on problems involving quantitative relationships:
     - ✓ *decontextualize* – to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents and
     - ✓ *contextualize* – to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.
   - use quantitative reasoning that entails habits of creating a coherent representation of the problem at hand:
     - ✓ considering the units involved,
     - ✓ attending to the meaning of quantities (not just how to compute them), and
     - ✓ knowing and flexibly using different properties of operations and objects.

Common Core State Standards: www.corestandards.org/Math
National Council of Supervisors of Mathematics: www.mathedleadership.org

Common Core Standards for Mathematical Practice

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students:

- understand and use stated assumptions, definitions, and previously established results in constructing arguments.
- make conjectures and build a logical progression of statements to explore the truth of their conjectures.
- analyze situations by breaking them into cases.
- recognize and use counterexamples.
- justify their conclusions, communicate them to others, and respond to the argument of others.
- reason inductively about data, making plausible arguments that take into account the context from which the data arose.
- compare the effectiveness of plausible arguments.
- distinguish correct logic or reasoning from that which is flawed and, if there is a flaw in an argument, explain what it is.
  - ✓ elementary students construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
  - ✓ later students learn to determine domains to which an argument applies.
- listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve arguments.

4. Model with mathematics.

Mathematically proficient students:

- apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
  - ✓ In early grades, this might be as simple as writing an addition equation to describe a situation.
  - ✓ In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
  - ✓ By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.
- apply what they know to make assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.
- identify important quantities in a practical situation.
- map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- analyze those relationships mathematically to draw conclusions.
- routinely interpret their mathematical results in the context of the situation.
- reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Common Core State Standards: www.corestandards.org/Math
National Council of Supervisors of Mathematics: www.mathedleadership.org

Common Core Standards for Mathematical Practice

5. Use appropriate tools strategically.

Mathematically proficient students:

- consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software.
- are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.
- High school students:
  - ✓ analyze graphs of functions and solutions generated using a graphing calculator.
  - ✓ detect possible errors by strategically using estimations and other mathematical knowledge.
  - ✓ when making mathematical models, know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.
- identify relevant external mathematical resources (e.g., digital website content) and use them to pose or solve problems.
- use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students:

- try to communicate precisely to others.
- try to use clear definitions in discussion with others and in their own reasoning.
- state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- carefully specify units of measure and label axes to clarify the correspondence with quantities in a problem.
- calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
  - ✓ In the elementary grades, students give carefully formulated explanations to each other.
  - ✓ By high school, students have learned to examine claims and make explicit use of definitions.

Common Core State Standards: www.corestandards.org/Math
National Council of Supervisors of Mathematics: www.mathedleadership.org

Common Core Standards for Mathematical Practice

7. Look for and make use of structure.

Mathematically proficient students:

- look closely to discern a pattern or structure.
  - ✓ Young students might notice that three and seven more is the same amount as seven and three more or they may sort a collection of shapes according to how many sides the shapes have.
  - ✓ Later, students will see 7 x 8 equals the well-remembered 7 x 5 +7 x 3, in preparation for the distributive property.
  - ✓ Older students, in the expression $x^2 + 9x + 14$, can see the 14 as 2 x 7 and the 9 as 2 +7.
  - ✓ They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.
- step back for an overview and shift perspective.
- see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects (e.g., They see $5 – 3(x\text{-}y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real number *x* and *y*).

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students:

- notice if calculations are repeated.
- look both for general methods and shortcuts.
  - ✓ Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeated decimal.
  - ✓ By paying attention to the calculation of slope as they repeatedly check whether the points are on the line through (1, 2) with a slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$.
  - ✓ Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead high school students to the general formula for the sum of a geometric series.
- maintain oversight of the process of solving a problem, while attending to the details.
- continually evaluate the reasonableness of their intermediate results.

Common Core State Standards: www.corestandards.org/Math
National Council of Supervisors of Mathematics: www.mathedleadership.org

Teacher and Student "Look Fors"

Teacher "Look Fors"	Student "Look Fors"
Establish productive classroom procedures and routines	Know and follow classroom procedures and routines
Foster the mathematical Habits of Mind and Habits of Interaction	Use Habits of Mind and Habits of Interaction without prompting
Promote a status-free classroom by calling on students equitably to answer or share ideas/reasoning	Rely on their own thinking and mathematical logic and do not defer to others based on status (status-free)
Engage students in worthwhile tasks regularly (Launch, Explore, Summarize)	Collaborate, discuss, and make sense of the mathematics with peers (experience productive disequilibrium)
Provide and honor Private Reasoning/Think Time	Honor Private Reasoning/Think Time by not calling out answers immediately
Scan the Room and not assist students immediately	Start the assignment promptly with limited assistance from the teacher
Use appropriate manipulatives, models, technology and other tools	Select and use appropriate manipulatives, models, technology and other tools
Ask questions to focus, assess, and advance students' mathematical thinking and reasoning	Share mathematical thinking, reasoning, and justifications
Confer with students individually, in small groups, and in whole groups	Confer with teacher and other students
Utilize productive math talk (Accountable Talk, A/B Dyads & Productive Talk Moves)	Listen to understand and share/compare their own and others' thinking
Select and Sequence students' mathematical ideas for discussion	Explain and justify their strategies and/or representations
Use Public Record/Anchor Chart to capture and highlight key strategies	Reference and use Public Record/Anchor Charts to understand strategies
Connect multiple representations of students' mathematical thinking	- Ask clarifying questions and respond to other students' ideas - Justify and defend their thinking by making connections
Use mistakes/stuck points and other teachable moments as opportunities for learning	Openly share their thinking and reasoning without fear of being incorrect
Provide timely and specific feedback to students	Articulate their current progress and what they know
Use frequent and varied opportunities to assess student learning to inform instructional decisions	Experience a variety of opportunities to show what they know

Framework Purpose and Structure:

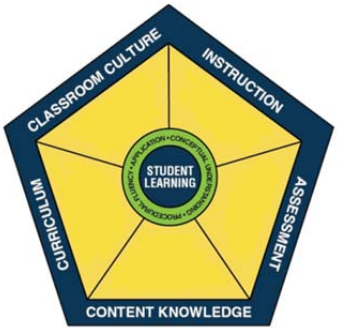
The <u>Teaching for Understanding Mathematics Framework</u> was developed to:

- **support teachers and administrators in improving student learning.**
- **provide common language and processes for mathematics instruction K-12.**
- **clearly define the characteristics of an effective mathematics classroom.**
- **provide a cohesive approach to professional learning and making instructional/curricular decisions.**

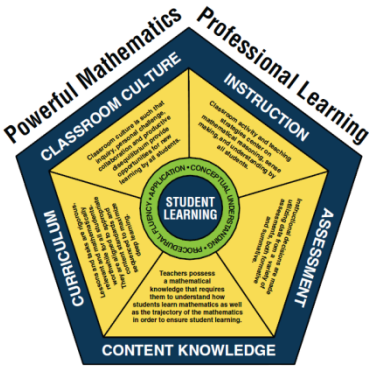
This Framework is research-based and is the product of collaborative contributions from a variety of stakeholders including teachers, administrators, researchers, and professional development providers. A description of how to interpret the visual for the Framework is provided below.

At the center of the Framework is the focus of all our work, **Student Learning**. This Framework is built upon the belief that all students **<u>can</u>** and **<u>must</u>** learn mathematics.

In order to maximize **Student Learning**, instruction must promote a balance of **Conceptual Understanding**, **Procedural Fluency**, and **Application**.



The components of an effective classroom are grouped into five domains: **Classroom Culture, Instruction, Assessment, Content Knowledge, and Curriculum**.

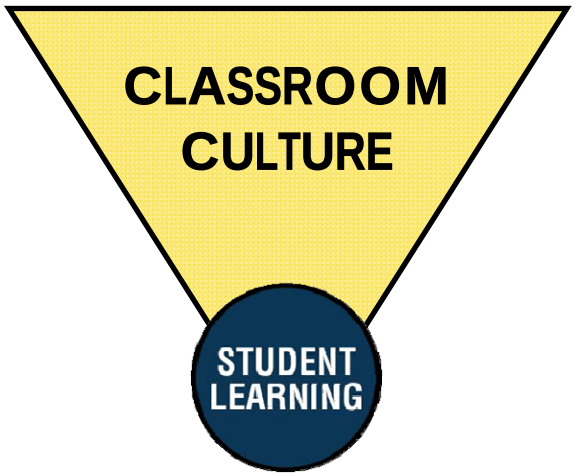


The **Core Ideas** are statements that clearly define each of the five domains.

The support document that accompanies the visual is designed to clearly define **Characteristics** of an effective mathematics classroom. The document also includes a wealth of planning and instructional tools that Hamilton County is defining as common best practices for K-12 mathematics instruction.

Domain Characteristics

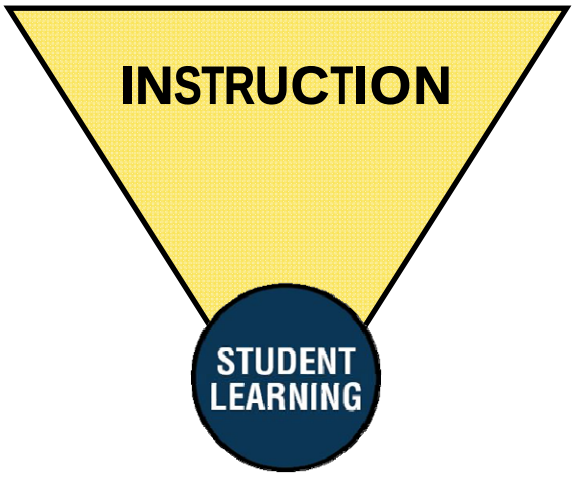
Classroom Culture	**14**
Instruction	**15**
Assessment	**16**
Content Knowledge	**17**
Curriculum	**18**



Core Idea: Classroom culture must be such that inquiry, personal challenge, collaboration and productive disequilibrium provide opportunities for new learning by all students.

Characteristics of a Productive Classroom Culture

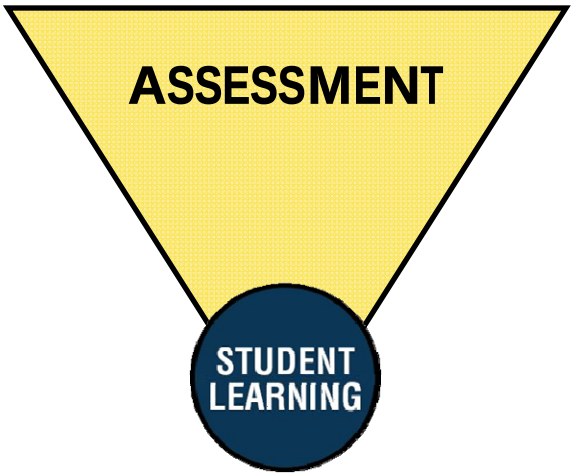
- All students are held to high expectations and provided opportunities to learn rigorous and challenging mathematics.
- Student and teacher interactions and structures promote a growth mindset. Intelligence is cultivated through effort and continued learning.
- The Habits of Mind are the established norms for how students approach mathematics. The Habits of Interaction are the established norms for how interaction occurs in the classroom.
- Students clearly understand classroom procedures and routines.
- Student and teacher interaction is respectful and conducive to learning.
- Students are individually and collectively accountable for the learning process.
- Students explain, question, critique and debate their mathematical thinking and reasoning and the reasoning of others.
- Students rely on their own thinking and mathematical logic to judge the correctness and usefulness of ideas. They do not defer to others based on status.
- Students experience productive disequilibrium that leads to conceptual understanding.
- Students view making mistakes and persevering through stuck points as a natural part of the learning process. They use mistakes and stuck points as opportunities for trying alternative strategies and examining contradictions.
- Students persevere in solving difficult and challenging problems.
- Students are encouraged to reflect upon their own thinking and ask questions to further their own learning.
- Students have autonomy in choosing and sharing their methods of solving problems.
- Students value Private Reasoning/Think Time for themselves and others.
- Students collaborate to solve problems.
- Students actively listen for understanding when others explain and justify.



Core Idea: Classroom activity and teaching strategies center on mathematical reasoning, sense making and understanding by all students.

Characteristics of Effective Learning

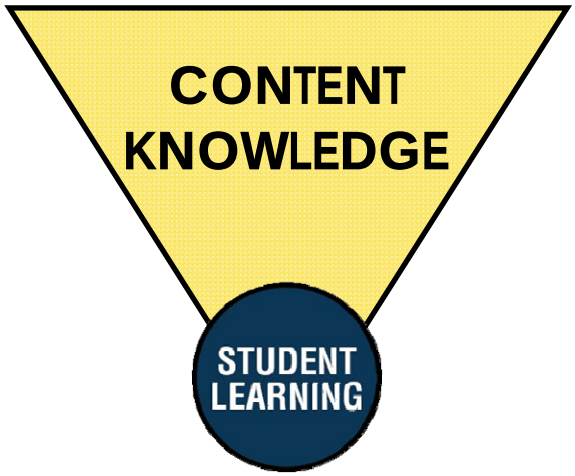
- Students believe they can learn and understand mathematics and will be supported in their efforts to accomplish this goal.
- Students activate their prior knowledge and experiences as they build new knowledge.
- Students look for and make use of structure, regularity, and repeated reasoning as they make sense of the mathematics.
- Students select and use models, manipulatives, technology, and other mathematical tools to make sense of ideas, solve problems, and invent procedures.
- Students approach problems in a variety of ways, using a variety of representations (visual, verbal, numerical, algebraic, graphical, and/or everyday life).
- Students formulate genuine questions, collect evidence, and form logical conclusions based on mathematical reasoning.
- Students explain and justify why and how their methods do or do not work.
- Students compare their strategies and results for mathematically significant similarities and differences.
- Student explanations include mathematical argumentation beyond a description or summary of procedures. An acceptable explanation is based on the mathematics, not the status of an individual.
- Students experience differentiated instruction based on the data collected from assessments.



Core Idea: Instructional decisions are made utilizing data from a variety of assessments, both formative and summative.

Characteristics of Effective Assessment

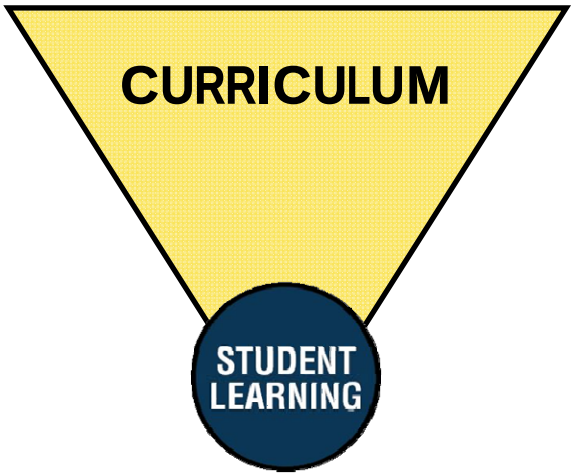
- Students are provided frequent opportunities to demonstrate their knowledge and skill through a variety of assessments.
- Students select and utilize appropriate technology and tools to assess their understanding.
- Students experience differentiated assessments.
- Students experience selected, open, and constructed-response assessment items similar to those on state and national assessments.
- Students experience opportunities to apply concepts, skills, reasoning, and problem-solving through performance-based assessments.
- Students experience opportunities to self-assess and reflect both on their own work and the ideas put forth by others.
- Students experience opportunities to define/formulate problems, explore possible solutions, and justify/defend their thinking.
- Students experience genuine questions that promote a deeper conceptual understanding.



Core Idea: Teachers possess a mathematical knowledge that requires them to understand how students learn mathematics as well as the trajectory of the mathematics in order to ensure student learning.

Characteristics of the Mathematical Knowledge Needed for Teaching

- Teachers possess both a deep content and pedagogical knowledge in mathematics in order to better anticipate and overcome student misconceptions.
- Teachers possess a deep understanding of mathematics: concepts, practices, principles, representations, technology, and applications.
- Teachers possess a deep understanding of the ways students learn mathematics and incorporate this knowledge in their instruction.
- Teachers implement methods that draw out and build upon student mathematical thinking.
- Teachers understand the trajectory of the content which enables them to effectively plan and make instructional decisions to impact important mathematical learning.
- Teachers pose questions and select tasks that foster important and long-lasting mathematical understanding.
- Teachers listen intently to students' thinking and respond based on mathematical implications and potential for extended learning.
- Teachers continually learn and grow with their students, remain current in the field, and engage in reflective practice.



Core Idea: Lessons and tasks are rigorous, relevant, and mathematically worthwhile for all students. They are aligned to appropriate content standards and sequenced to maximize deep learning.

Characteristics of Effective Curriculum

- Students experience rigorous lessons and tasks that are complex, ambiguous, and personally challenging.
- Students are given opportunities to experience mathematically productive disequilibrium.
- Students experience lessons and tasks that connect mathematical ideas to relevant everyday life and/or other mathematics contexts.
- Students experience lessons and tasks that are mathematically worthwhile: focus on significant mathematics, offer appropriate challenge, and allow for multiple perspectives.
- Students experience lessons and tasks that are directly aligned to the appropriate grade level, state, and national standards.
- Students experience lessons and tasks that are selected and sequenced to maximize student learning.
- Students have access to resources that will promote deep understanding of mathematics.
- Students use multiple representations, a variety of manipulatives, technology, and other mathematical tools to solve problems.